

PERSISTENT SUPERCURRENTS IN A PLANAR NON-RELATIVISTIC CHIRAL FLUID

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Abstract

We study the possible stationary persistent supercurrents flowing on a cylindrical sample supporting a two-dimensional charged fluid. The internal dynamics of the fluid is obtained by means of an effective theory in which the fluid self-interacts through a $U(1)$ gauge field. We find that the presence of persistent supercurrents depends on what kind of gauge field it is. In particular the current is zero if it is a Maxwell gauge field, and it is maximal if it is a Chern-Simons gauge field. There is an intermediate behaviour in presence of both Maxwell and Chern-Simons term. Therefore it appears that persistent supercurrents are possible only if the fluid is chiral.

1. Introduction and summary.

In this paper we study the possibility that stationary supercurrents can flow, like in a superconductor, in a planar non-relativistic fluid which looks incompressible in the sense that all the excitations have an energy gap and thus there are no gapless compressional modes.

This fluid is described by the most general effective action in two space and one time dimensions including $U(1)$ gauge fields. As it is well known (references [1] [2]), in $2+1$ dimensions the generic action with at most two derivatives for the gauge fields includes both a Maxwell-like term and a parity and time reversal violating Chern-Simons term. A gauge theory of that kind has been introduced long ago and extensively studied also for possible relations with Chern-Simons theory describing fractional statistics particles and rather speculative applications to high T_c superconductors (see reference [3] for a general introduction).

Here we study an effective action where the $U(1)$ gauge field is coupled to a non-relativistic matter field. This effective theory was studied for the pure Chern-Simons case in reference [4]. The possible vortex excitations of this system for the Chern-Simons case and for the Maxwell case were discussed in reference [5], and later also in [6]. (An effective theory formally similar to the one we investigate here, in the Chern-Simons limit, was used to describe the Fractional Quantum Hall Effect in references [7], [8], [9]. Notice that the physical system we consider is completely different from the quantum Hall setting, in particular there is no uniform magnetic field, provided by a given external device).

One can easily derive the spectrum (see section 2) and find that the possible excitations have an energy gap. Thus, this fluid does not appear similar to a standard superconductor (nor to the standard anyonic superconductors [10], [11], [12], [13], [14]), which possesses a gapless excitation that plays the role of the Higgs field when coupled to the electromagnetic field. Nevertheless, we specifically consider the possibility of stationary currents flowing around in an idealized cylindrical sample, as the one of figure 1a, much like a supercurrent in a superconducting solenoid. We consider the cylinder made of a pile of many two dimensional annuli, on which the fluid's current flows around. We also consider the fluid to be charged and coupled to ordinary three-dimensional electromagnetism, and we study in particular the magnetic field produced by the possible supercurrents.

We find in section 3 that the possibility of supercurrents depends very much on the relative weight of the Maxwell term as compared to the Chern-Simons term, in the two-dimensional (gauge) dynamics. An important role is played by the electrostatic force which arises when there is a charge fluctuation (we assume that our system, fluid plus background, is overall electrically neutral. However we assume that the background is rigid, and thus a fluctuation in the density of the charged fluid gives rise to an electrostatic field). In the limit where the Maxwell term dominates, no currents are possible in the bulk of the sample – in this case the currents are concentrated at the edges and are small. Thus in this limit our sample does not work like a superconducting solenoid.

On the contrary, in the limit where the Chern-Simons term dominates, the current can flow freely like in a superconductor and our system behaves like a superconducting solenoid (of the same thickness, of course), the resulting magnetic field and flux being the same as

the ones due to a supercurrent for an ordinary superconductor. In the intermediate cases we find intermediate behaviours.

Since there is a gap in the spectrum, the flow of the current is protected against external disturbances, in the sense that as far as the energy carried by the external sources is less than the gap, no interaction is possible.

Thus, for a sizable Chern-Simons term, we find that, despite being non-standard, our fluid has some properties similar to standard superconductors. Actually we studied [4] already other properties of the non-relativistic fluid coupled to a three-dimensional $U(1)$ gauge field dominated by the Chern-Simons term and we found interesting results showing similarities with superconductors. In particular, we studied the possible penetration of an external magnetic field (Meissner effect) in the bulk of an idealized sample of many two-dimensional layers where the fluid lies. We found an intermediate behaviour, namely if the magnetic field is orthogonal to the layers (that is, in a configuration similar to the one studied here) the system behaves in the same way as a type II superconductor, whereas if the the field is parallel to the layers it can penetrate much more easily, going to zero inside with an inverse power law, rather than with an exponential law like in the standard case.

Thus, putting together the previous results and the ones presented in this paper, we can conclude that a chiral fluid (chirality being related to the parity breaking Chern-Simons term) shows interesting properties which could qualify it, in some respect, as a kind of non-standard superconductor.

2. The effective lagrangian.

Here we study a non-relativistic theory in 2+1 dimensions described by the following lagrangian:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}g_E (\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}g_B (\epsilon_{ij}\partial_i A_j)^2 - \\ & -\alpha\epsilon_{\lambda\mu\nu}A_\lambda\partial_\mu A_\nu + \frac{1}{2m}\phi^\dagger\vec{D}^2\phi + \frac{i}{2}\left(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi\right) + A_0(|\phi|^2 - \rho_0) - V(|\phi|) . \end{aligned} \quad (2.1)$$

Here $\phi(\vec{x}, t)$ is a complex field which plays the role of order parameter and is related to the density by the relation:

$$\rho(\vec{x}, t) = |\phi(\vec{x}, t)|^2 . \quad (2.2)$$

The covariant derivative is $D_i = \partial_i - iA_i$. Notice that A_0 is coupled to $\delta\rho = \rho - \rho_0$, and that ρ_0 is the average density representing the neutralizing background necessary for the consistency of the theory. Since for conservation of the total number of particles the integral over all the surface of $\delta\rho$ is zero, we can write it as the divergence of some vector field $\vec{u}(\vec{x})$ which we choose to be irrotational $\vec{\nabla} \wedge \vec{u} = 0$:

$$\delta\rho = \vec{\nabla} \cdot \vec{u} \quad (2.3)$$

Equation (2.1) is the most general non-relativistic gauge invariant lagrangian with minimal coupling to a vector potential, once the maximum number of two derivatives is required.

First note that beside the Maxwell term there is also a Chern-Simons term with coupling constant α . Notice also that the coupling constants for the electric and the magnetic part of the Maxwell term are different. They are equal only if the theory is Lorentz invariant, that is considering a relativistic theory (here we draw the reader's attention to the fact that in 2+1 dimensions these coupling constants have the dimension of a length and α is dimensionless). In the following, for definiteness, we take $V(|\phi|) = \frac{1}{2}\lambda(|\phi|^2 - \rho_0)^2$. Notice in particular that varying (2.1) with respect to A_i , A_0 one gets (in the gauge $\vec{\nabla} \cdot \vec{A} = 0$):

$$g_E \partial_0 (\partial_0 A_i - \partial_i A_0) + g_B \epsilon_{ij} \partial_j B = J_i - 2\alpha \epsilon_{ij} (\partial_j A_0 - \partial_0 A_j) \quad (2.4)$$

$$g_E \triangle A_0 = \delta\rho - 2\alpha B. \quad (2.5)$$

We have defined:

$$\delta\rho = \rho - \rho_0 \quad B = \epsilon_{ij} \partial_i A_j \quad J_i = \frac{1}{2mi} (\phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \phi - 2i A_i \phi^\dagger \phi) \quad (2.6)$$

Thus for $\alpha=0$ we recover the two-dimensional Maxwell equations (though with two different couplings). For $g_E=0$, on the other hand, from equation (2.5) we get $\delta\rho=2\alpha B$ which is the usual Chern-Simons constraint relating the field strength to the matter density. Later on we will also consider the fluid to be electrically charged and thus coupled to the true, three-dimensional electromagnetic field.

2.1. The spectrum.

Now we will study the energy spectrum of the small field perturbations of our Maxwell-Chern-Simons theory.

We take the following parameterization:

$$\phi = f e^{i\theta}. \quad (2.7)$$

and rewrite lagrangian (2.1) keeping only quadratic terms in the fields. With this choice, and in the gauge $\vec{\nabla} \cdot \vec{A} = 0$, equation (2.1) becomes (in this gauge one can put $A_i = \epsilon_{ij} \partial_j \psi$ and see that $\int d^2x \epsilon_{ij} A_i \partial_0 A_j = 0$):

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} g_E \left[(\partial_0 A_i)^2 + (\partial_i A_0)^2 \right] - \frac{1}{2} g_B (\partial_i A_j)^2 + \\ & - 2\alpha \epsilon_{ij} A_0 \partial_i A_j + \frac{1}{2m} \left[-\frac{1}{4\rho_0} (\partial_i \delta\rho)^2 - \rho_0 (\partial_i \theta)^2 - \rho_0 A_i^2 \right] + \theta \partial_0 (\delta\rho) + A_0 \delta\rho - \frac{1}{2} \lambda (\delta\rho)^2. \end{aligned} \quad (2.8)$$

From this lagrangian we get the following equations:

$$\frac{\delta \mathcal{L}}{\delta A_0} = -g_E \triangle A_0 - 2\alpha \epsilon_{ij} \partial_i A_j + \delta\rho = 0 \quad (2.9)$$

$$\frac{\delta \mathcal{L}}{\delta A_i} = -g_E \partial_0^2 A_i + g_B \triangle A_i - 2\alpha \epsilon_{ij} \partial_j A_0 - \frac{\rho_0}{m} A_i = 0 \quad (2.10)$$

$$\frac{\delta \mathcal{L}}{\delta \theta} = \frac{\rho_0}{m} \triangle \theta + \partial_0(\delta \rho) = 0 \quad (2.11)$$

$$\frac{\delta \mathcal{L}}{\delta(\delta \rho)} = \frac{1}{4m\rho_0} \triangle \delta \rho - \partial_0 \theta + A_0 - \lambda \delta \rho = 0 . \quad (2.12)$$

Multiplying (2.10) by $\epsilon_{ki} \partial_k$ we get:

$$-g_E \partial_0^2 B + g_B \triangle B + 2\alpha \triangle A_0 - \frac{\rho_0}{m} B = 0 . \quad (2.13)$$

Now multiplying equation (2.11) by ∂_0 and equation (2.12) by \triangle we get

$$\triangle \dot{\theta} = -\frac{m}{\rho_0} \partial_0^2(\delta \rho) , \quad (2.14)$$

and

$$\frac{1}{4m\rho_0} \triangle^2 \delta \rho - \triangle \dot{\theta} + \triangle A_0 - \lambda \triangle \delta \rho = 0 . \quad (2.15)$$

We can now eliminate θ and A_0 from equations (2.13) and (2.15) by using (2.9) and (2.14). By taking $B = (B_0 e^{i(\mathcal{E}t - \vec{p} \cdot \vec{x})} + c.c.)$ and $\delta \rho = (\delta \rho_0 e^{i(\mathcal{E}t - \vec{p} \cdot \vec{x})} + c.c.)$ we thus obtain:

$$\mathcal{E}^2 B_0 - \left(\frac{4\alpha^2}{g_E^2} + \frac{\rho_0}{mg_E} \right) B_0 + \frac{2\alpha}{g_E^2} \delta \rho_0 - \frac{g_B}{g_E} (p_x^2 + p_y^2) B_0 = 0 \quad (2.16)$$

$$\mathcal{E}^2 \delta \rho_0 - \frac{\rho_0}{mg_E} \delta \rho_0 + \frac{2\alpha \rho_0}{mg_E} B_0 - \frac{\lambda \rho_0}{m} (p_x^2 + p_y^2) \delta \rho_0 - \frac{1}{4m^2} (p_x^2 + p_y^2)^2 \delta \rho_0 = 0 . \quad (2.17)$$

Therefore the values of the energy for $\vec{p}=0$ are the eigenvalues of the above system:

$$\mathcal{E}_{1,2}^2 = \frac{2\alpha^2}{g_E^2} + \frac{\rho_0}{mg_E} \pm \frac{2\alpha^2}{g_E^2} \sqrt{1 + \frac{g_E \rho_0}{m\alpha^2}} . \quad (2.18)$$

Notice that $\mathcal{E}_{1,2}^2$ are both positive. If $\alpha \rightarrow 0$, that is if there is no Chern-Simons term, the two eigenvalues are equal:

$$\mathcal{E}_1 = \mathcal{E}_2 = \sqrt{\frac{\rho_0}{mg_E}} . \quad (2.19)$$

If, on the other hand, we take the opposite limit $g_E \rightarrow 0$ then only one of the eigenvalues remains finite whereas the other one diverges:

$$\begin{aligned} \mathcal{E}_1 &= \frac{2|\alpha|}{g_E} \rightarrow \infty \\ \mathcal{E}_2 &= \frac{\rho_0}{2m|\alpha|} . \end{aligned} \quad (2.20)$$

That is, in general there are two propagating degrees of freedom each with its energy gap. In pure Maxwell theory the energy gaps are degenerate. This degeneracy is resolved by adding the Chern-Simons term. If there is only the Chern-Simons term the theory possesses only one degree of freedom with an energy gap, the other one being frozen. This value of the energy gap for the pure Chern-Simons case has already been derived in reference [4].

3. Stationary currents.

In this section we study the possibility of stationary currents. We would like to study a configuration that, although idealized, is near to a setting relevant for a realistic device which could carry superconducting currents. Therefore we take the surface where our two-dimensional fluid lies to be an annulus of radii $r_2 > r_1$ such that $r_2 - r_1 = L$. We consider a pile of many of these annuli separated by a spacing d , see figure 1a. We assume that $L/r_1 \ll 1$, but still L is macroscopic, that is $L/d \gg 1$ where d is any length of the order of few atomic distances such as the inter-annuli spacing. Furthermore, the total length L_z of the cylinder is supposed to be very large as compared to the radii $r_{1,2}$. We study the possible configuration where the fluid's current flows around in each annulus, see figure 1b, and it is uniform with respect to z , that is it is the same in each annulus. Thus our cylindrical configuration will act as a solenoid. Later we will compute the three-dimensional true magnetic field inside the solenoid, due to the current.

Figure 1

From the two-dimensional point of view the annulus is equivalent to a strip which is finite in the x direction, $-L/2 \leq x \leq L/2$, and periodic in the y direction, $0 \leq y \leq 2\pi R_M$, see figure 1c (with $R_M = \frac{r_1+r_2}{2}$, remember that in this approximation $L/R_M \ll 1$). We consider a possible current induced by a quantized phase for the matter field, as for a vortex:

$$\phi = |\phi|e^{ipy} \quad (3.1)$$

where $p = \frac{n}{R_M}$, n integer, is fixed. The current flows around the cylinder as in figure 1b. Notice that we consider a configuration with cylindrical symmetry, therefore we take (in the gauge $\vec{\nabla} \cdot \vec{A} = 0$):

$$\begin{aligned} \vec{A} &= (A_x, A_y) = (0, A(x)) & B &= \partial_x A \\ \vec{u} &= (u(x), 0) & \delta\rho(x) &= \partial_x u(x) & |\phi| &= \sqrt{\delta\rho(x) + \rho_0} . \end{aligned} \quad (3.2)$$

As we said we assume uniformity in the z direction and we consider the limit where the cylinder's length L_z is very large as compared to R_M , so we can write the hamiltonian of the model as:

$$\begin{aligned} \frac{H}{L_z} = \int dxdy \left\{ \frac{1}{2} g_B B^2 - \frac{1}{2m} \phi^\dagger \vec{D}^2 \phi + V(|\phi|) \right\} + \\ + \frac{1}{2g_E} (\delta\rho - 2\alpha B) * \frac{1}{(-\Delta)} * (\delta\rho - 2\alpha B) . \end{aligned} \quad (3.3)$$

Here the last term comes from integrating out the auxiliary field A_0 and the symbol $*$ means convolution.

It is convenient to express this hamiltonian as the integral of a local hamiltonian density. To this aim we rewrite the last term of (3.3) in terms of the two-dimensional “electrostatic” field \vec{E} determined by the equation:

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{g_E} (\delta\rho - 2\alpha B) \quad (3.4)$$

and imposing boundary conditions such as to eliminate possible zero modes which do not contribute to the hamiltonian (3.3). This amounts to requiring (in our configuration with cylindrical symmetry we have uniformity in the y coordinate, therefore $\vec{E} = (E(x), 0)$):

$$E(x < -L/2) = E(x > L/2) = 0 . \quad (3.5)$$

The hamiltonian can then be rewritten as

$$\frac{H}{L_z} = \int dxdy \left\{ \frac{1}{2} g_E E^2 + \frac{1}{2} g_B B^2 - \frac{1}{2m} \phi^\dagger \vec{D}^2 \phi + V(|\phi|) \right\} \quad (3.6)$$

We have to fix completely the boundary conditions. We assume that:

$$A(x < -L/2) = A(x > L/2) = 0 \quad (3.7)$$

that is, we assume A , which represents in general a propagating excitation of our fluid, to be zero outside the sample i.e. where there is no fluid. Notice that, by subtracting the irrelevant constant $c_0 = -\frac{1}{2} \lambda \rho_0 N$ where N is the fixed total particles' number $N = \int |\phi|^2$, we can rewrite $\int V = \frac{1}{2} \lambda \int |\phi|^4$ which vanishes outside the sample.

One can also say that, by definition, $\delta\rho = 0$ outside the sample. Since the fluctuation of the particles' total number in a given domain is the flux of \vec{u} through its boundary, and the particles' total number is fixed, the boundary conditions for u are

$$u(x < -L/2) = u(x > L/2) = 0 \quad (3.8)$$

Thus, keeping into account (3.5), we see that the hamiltonian density vanishes outside the sample, consistently with the fact that it describes the dynamics of the fluid which is confined in the strip.

We can check that the boundary condition (3.5) is the one which correctly ensures that the hamiltonian (3.6) implements the Chern-Simons dynamics in the limit $g_E \rightarrow 0$. In fact, the solution of equation (3.4), with the boundary conditions (3.5), (3.7) and (3.8) gives:

$$E(x) = -\frac{1}{g_E} \int_{-\frac{L}{2}}^x dx' \delta\rho(x') + \frac{2\alpha}{g_E} A(x) = -\frac{1}{g_E} [u(x) - 2\alpha A(x)] . \quad (3.9)$$

Thus we see that, since finite energy requires $g_E E^2$ to be bounded, one recovers in the limit $g_E \rightarrow 0$:

$$A(x) = \frac{1}{2\alpha} u(x) \quad (3.10)$$

which is indeed the solution of the Chern-Simons constraint $B = \frac{1}{2\alpha} \delta\rho$ with the boundary conditions of (3.7) and (3.8).

To this hamiltonian we now add one extra term which plays an essential role in what follows. This term represents the true, three-dimensional electrostatic interaction between the fluctuations of the charged matter. Of course, the whole system, fluid plus background, is overall electrically neutral. But if the background is rigid, as we assume here, a local density fluctuation of the charged fluid gives rise to electrostatic forces. This yields a *real* electric field which obeys to the three-dimensional Maxwell equation:

$$\vec{\nabla} \cdot \vec{E}^{em} = e\delta\rho^{(3)} , \quad (3.11)$$

where $\delta\rho^{(3)} = \delta\rho/d$ is the three dimensional density and e is the electric charge. (for other effects of this electrostatic contribution we refer to [4]). We can then compute this electrostatic contribution. The three-dimensional Maxwell equation can be written as (in our uniform in y and in z configuration):

$$\partial_x E^{em}(x) = \frac{e}{d} \delta\rho = \frac{e}{d} \partial_x u . \quad (3.12)$$

This equation can be integrated yielding:

$$E^{em} = \frac{e}{d} u \quad (3.13)$$

Therefore the electrostatic contribution to the hamiltonian is

$$\frac{d}{2} (\vec{E}^{em})^2 = \frac{e^2}{2d} u^2 \quad (3.14)$$

Thus, after taking everything into account, the hamiltonian of the system can be written as:

$$\begin{aligned} \frac{H}{2\pi R_M L_z} = \int dx \left\{ \frac{1}{2g_E} (u - 2\alpha A)^2 + \frac{1}{2} g_B (\partial_x A)^2 + \frac{1}{2m} (p - A)^2 \rho + \frac{1}{2m} (\partial_x |\phi|)^2 + \right. \\ \left. + \frac{1}{2} \lambda (\partial_x u)^2 + \frac{e^2}{2d} u^2 \right\} \end{aligned} \quad (3.15)$$

Before going on it is maybe worth to stress again the difference between $E(x)$ and $E^{em}(x)$. The first one is the two-dimensional “electric field” coming from the internal dynamics of the fluid. The second one is the actual, real, three-dimensional electric field due to electrostatic interaction of the charged matter.

3.1. A heuristic analysis.

We perform a heuristic analysis by considering a situation where the matter is uniformly distributed and there is no “magnetic field”, that is:

$$\delta\rho = 0 \quad B = \partial_x A = 0 . \quad (3.16)$$

This, in particular, implies $\rho = \rho_0$ and $\partial_x |\phi| = 0$. In this case the variation of the hamiltonian (3.15) gives the equations

$$\begin{aligned} \frac{\delta}{\delta A} \left(\frac{H}{2\pi R_M L_z} \right) &= \frac{2\alpha}{g_E} (2\alpha A - u) + \frac{\rho_0}{m} (A - p) = 0 \\ \frac{\delta}{\delta u} \left(\frac{H}{2\pi R_M L_z} \right) &= \frac{1}{g_E} (u - 2\alpha A) + \frac{e^2}{d} u = 0 . \end{aligned} \quad (3.17)$$

The solution of these equations is (in this heuristic analysis we forget the boundary conditions):

$$\begin{aligned} A_* &= \frac{\frac{\rho_0}{m}}{\frac{\rho_0}{m} + \frac{4\alpha^2 e^2}{d + e^2 g_E}} p \\ u_* &= \frac{2\alpha \rho_0 d}{\rho_0 (d + e^2 g_E) + 4m\alpha^2 e^2} p . \end{aligned} \quad (3.18)$$

This corresponds to the current density:

$$J = \frac{\rho_0}{m} \frac{\frac{4\alpha^2 e^2}{d + e^2 g_E}}{\frac{\rho_0}{m} + \frac{4\alpha^2 e^2}{d + e^2 g_E}} p . \quad (3.19)$$

Notice that if $e^2 = 0$, that is if there is no electrostatic effect, the density current is zero. Notice also that in the Maxwell limit ($\alpha \rightarrow 0$ or $g_E \rightarrow \infty$) the current density is still zero. Therefore the current density is different from zero only if there is some Chern-Simons amount in the fluid’s dynamics. In particular for the case of pure Chern-Simons, with $\alpha \rightarrow \infty$, we recover the maximum value for the current:

$$J \xrightarrow{g_E \rightarrow 0} \frac{\rho_0}{m} \frac{4m\alpha^2 e^2}{\rho_0 d + 4m\alpha^2 e^2} p \xrightarrow{\alpha \rightarrow \infty} \frac{\rho_0}{m} p . \quad (3.20)$$

3.2. A more detailed analysis.

The assumptions of the previous section are physically too restrictive because the actual physical situation can have $\delta\rho$ and B different from zero. Nevertheless in this section we show that a more accurate computation leads substantially to the same results, apart from a correction which is different from zero only in regions very close to the sample's borders.

In order to keep the discussion reasonably transparent, we make the simplifying assumptions that $\rho_0 \gg |\delta\rho|$ and that $1/m \ll \lambda$. With these assumptions we can write:

$$\rho(p - A)^2 \simeq \rho_0(p - A)^2 \quad \frac{1}{2m}(\partial_x|\phi|)^2 = \frac{1}{8m\rho}(\partial_x\delta\rho)^2 \simeq 0. \quad (3.21)$$

We stress that our results would hold also in general. In fact, the equilibrium configuration of the fluid, far from the borders, will be the one of section 3.1. What happens in details near to the borders will depend on the details of the hamiltonian, and we have chosen to present the simplified discussion based on (3.21).

In this way the field equations now read:

$$\begin{aligned} \frac{\delta}{\delta A} \left(\frac{H}{2\pi R_M L_z} \right) &= -\partial_x^2 A + \frac{2\alpha}{g_E g_B} (2\alpha A - u) + \frac{\rho_0}{mg_B} (A - p) = 0 \\ \frac{\delta}{\delta u} \left(\frac{H}{2\pi R_M L_z} \right) &= -\partial_x^2 u + \frac{1}{\lambda g_E} (u - 2\alpha A) + \frac{e^2}{\lambda d} u = 0. \end{aligned} \quad (3.22)$$

These equations can be rewritten in matricial form as follows:

$$(-\mathbf{1}\partial_x^2 + M)\Psi = \Phi \quad (3.23)$$

where $\mathbf{1}$ is the unit two by two matrix, and

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{4\alpha^2}{g_E g_B} + \frac{\rho_0}{mg_B} & -\frac{2\alpha}{g_E g_B} \\ -\frac{2\alpha}{g_E \lambda} & \frac{1}{g_E \lambda} + \frac{e^2}{\lambda d} \end{pmatrix} \quad \Psi = \begin{pmatrix} A \\ u \end{pmatrix} \quad \Phi = \begin{pmatrix} \frac{\rho_0}{g_B m} p \\ 0 \end{pmatrix}. \quad (3.24)$$

Note that $\det M \neq 0$.

The general solution of equation (3.23) can be written as

$$\Psi(x) = \Psi_0 + \Psi_* \quad (3.25)$$

where

$$\Psi_* = M^{-1}\Phi = \begin{pmatrix} A_* \\ u_* \end{pmatrix} \quad (3.26)$$

is one particular solution and A_* and u_* are given by equation (3.18). Ψ_0 is the general solution of the homogeneous equation $(-\mathbf{1}\partial_x^2 + M)\Psi = 0$. The boundary conditions discussed in section 3 are $\Psi(-L/2) = \Psi(L/2) = 0$. Therefore we can write:

$$\begin{pmatrix} A(x) \\ u(x) \end{pmatrix} = \frac{\cosh(\mu_1 x)}{\cosh(\mu_1 L/2)} \begin{pmatrix} A_1 \\ u_1 \end{pmatrix} + \frac{\cosh(\mu_2 x)}{\cosh(\mu_2 L/2)} \begin{pmatrix} A_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} A_* \\ u_* \end{pmatrix} \quad (3.27)$$

where

$$\mu_{1,2}^2 = \frac{1}{2} \left(a + d \pm \sqrt{(a-d)^2 + 4bc} \right) \quad (3.28)$$

are the eigenvalues of the matrix M , and

$$u_{1,2} = \frac{\mu_{1,2}^2 - a}{b} A_{1,2} . \quad (3.29)$$

One can verify that $\mu_{1,2}^2$ are real positive. Let us mention their values in some limiting cases. For the case of pure Maxwell, i.e. $\alpha=0$, the fields $A(x)$ and $u(x)$ decouple and the two eigenvalues are

$$\mu_1^2 = \frac{\rho_0}{g_B m} \quad (\text{relevant for } A) \quad \mu_2^2 = \frac{1}{\lambda g_E} + \frac{e^2}{\lambda d} \quad (\text{relevant for } u) . \quad (3.30)$$

In the opposite limit $g_E \rightarrow 0$ the relevant field configuration is $u=2\alpha A$, and this is reflected in the eigenvalues which turn out to be

$$\mu_1^2 \rightarrow \infty \quad \mu_2^2 = \frac{\frac{\rho_0}{m} + \frac{4e^2\alpha^2}{d}}{g_B + 4\alpha^2\lambda} . \quad (3.31)$$

Imposing the boundary conditions, we determine $A_{1,2}$ by:

$$\begin{pmatrix} \frac{1}{b} & \frac{1}{b} \\ \frac{\mu_1^2 - a}{b} & \frac{\mu_2^2 - a}{b} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = - \begin{pmatrix} A_* \\ u_* \end{pmatrix} . \quad (3.32)$$

Notice that the two eigenvalues can be equal only for $\frac{\alpha}{g_E}=0$ and $\frac{\rho_0}{mg_B} = \frac{1}{g_E\lambda} + \frac{e^2}{\lambda d}$ in which case M is proportional to the identity, and in (3.27) we can write $A_1 = A_2$ and $u_1 = u_2$ with no relation between A_1 and u_1 . The boundary conditions give in this case $A_1 = -A_*$ and $u_1 = -u_*$.

Since $\mu_{1,2}^2$ are expressed in terms of the microscopic parameters of the hamiltonian, we have $L \cdot \mu_{1,2} \gg 1$. Thus, $A(x) = A_*$ and $u(x) = u_*$ apart from very near to the edges, that is for $|x| \simeq L/2$, and therefore we recover the solution found in the previous heuristic analysis.

3.3. Magnetic field.

In this subsection we compute the actual magnetic field present inside the cylinder (see figure 1a) due to the supercurrent computed above.

It is clear that we have to take now a three-dimensional view-point. Our three-dimensional system is now the cylinder embedded in the three-dimensional space. Therefore we will use the cylindrical coordinates (r, φ, z) as in figure 1a. We remind that we consider $L \ll r_{1,2}$ so that the cylinder can be thought of zero thickness and we take its radius to be $R_M = \frac{r_1 + r_2}{2}$.

The three-dimensional density of the fluid can be written in this approximation as $\rho^{(3)} = \frac{\rho_0}{d} L \cdot \delta(r - R_M)$.

Furthermore we suppose $L_z \gg R_M$ so that it is possible to keep far from the cylinder's edges and therefore to disregard the edge effects.

With these definitions the three-dimensional current density flowing all round the cylinder is:

$$J_\varphi = \frac{e\rho_0 L}{md} [p - eA_\varphi^{em}(r) - A] \delta(r - R_M) . \quad (3.33)$$

Here $A_\varphi^{em}(r)$ is the electromagnetic vector potential describing the magnetic field produced inside the cylinder. Since we limit ourselves to the region far from the edges of the cylinder, we can assume A_φ^{em} independent of z .

A is the (two-dimensional) vector potential describing the fluid's dynamics on the strip considered in the previous section.

We can reproduce all the computation of the previous section taking into account for A_φ^{em} simply with the substitution $p \rightarrow p - eA_\varphi^{em}$. Therefore we now have, instead of (3.18):

$$A = \frac{\frac{\rho_0}{m}}{\frac{\rho_0}{m} + \frac{4\alpha^2 e^2}{d + e^2 g_E}} (p - eA_\varphi^{em}) \quad (3.34)$$

We have neglected the deformation of A very near to the annuli's edges, as discussed in the previous section. Now we use this expression for A to solve Maxwell's equation $\vec{\nabla} \wedge \vec{B}^{em} = \vec{J}$ inside the cylinder (see reference [15]). The solution of this equation is:

$$A_\varphi^{em} = \frac{e\rho_0^{em}}{m} p \frac{1}{1 + \frac{e^2 \rho_0^{em} R_M}{2m}} \left[\frac{r}{2} \Theta(R_M - r) + \frac{R_M^2}{2r} \Theta(r - R_M) \right] . \quad (3.35)$$

where:

$$\rho_0^{em} = \frac{\frac{4e^2 \alpha^2}{d + e^2 g_E}}{\frac{\rho_0}{m} + \frac{4e^2 \alpha^2}{d + e^2 g_E}} \cdot \rho_0 \frac{L}{d} . \quad (3.36)$$

Here Θ is the step function. This yields the current density:

$$J_\varphi = \frac{e\rho_0^{em}}{m} \frac{p}{1 + \frac{e^2 \rho_0^{em} R_M}{2m}} \delta(r - R_M) , \quad (3.37)$$

We can also compute the magnetic field inside the cylinder using the cylindrical coordinate relation $B_z^{em} = \frac{1}{r} \frac{d}{dr} (r A_\varphi^{em})$:

$$B_z^{em} = \frac{e\rho_0^{em}}{m} \frac{p}{1 + \frac{e^2 \rho_0^{em} R_M}{2m}} \Theta(R_M - r) . \quad (3.38)$$

This result is the same as the one of a solenoid in which the current density (3.37) flows. We can also compute the flux of the magnetic field in the cylinder:

$$\Phi(B^{em}) = \pi R_M^2 B_z^{em} = \pi R_M^2 \frac{e\rho_0^{em}}{m} \frac{1}{1 + \frac{e^2 \rho_0^{em} R_M}{2m}} p . \quad (3.39)$$

Remembering that $p = \frac{n}{R_M}$, we see that for $e\rho_0^{em} R_M^2 \rightarrow \infty$ we get

$$\Phi(B^{em}) \longrightarrow \frac{2n\pi}{e} . \quad (3.40)$$

That is, the total flux is quantized in the above limit, as expected for a vortex-like current. We stress that, as we said, the result for the magnetic field and flux is exactly the same as it would have been obtained for an ordinary superconductor flowing on a cylindrical surface, ρ_0^{em} being its surface density. (It is a general fact for any superconducting current flowing on a cylindrical surface, that the flux quantization is strictly speaking obtained just in the above limit).

Note that ρ_0^{em} vanishes in the Maxwell limit $g_E \rightarrow \infty$ (or $\alpha = 0$) and $\rho_0^{em} = \rho_0 \frac{L}{d}$ for $\alpha \rightarrow \infty$. Therefore we see again that, in order that our idealized device could work as a superconducting solenoid, there cannot be only a Maxwell dynamics, but the chiral Chern-Simons term must play an essential role.

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Figure caption

Figure 1.

- a) The cylinder as a pile of many annuli
- b) A single annulus
- c) The annulus as a periodic strip

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